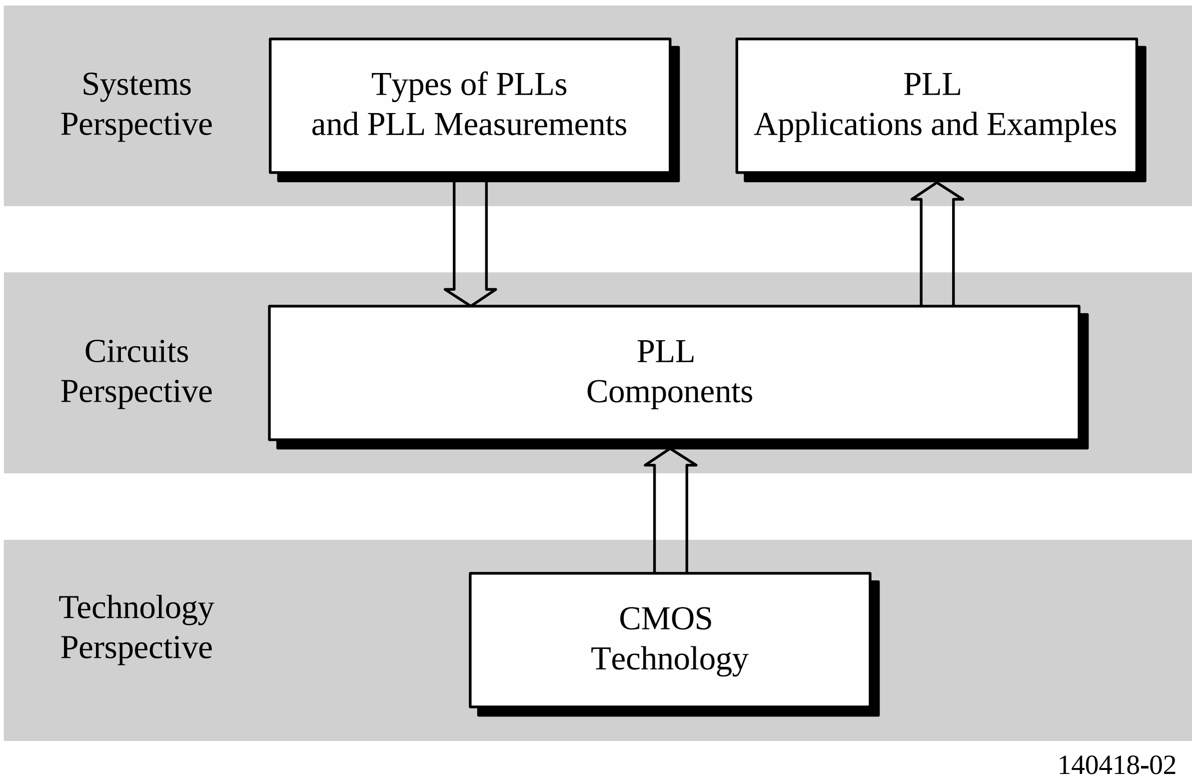
LECTURE 2 – CMOS PHASE LOCKED LOOPS

Topics

- Locked state of the LPLL
- Order of the LPLL

Organization:



LOCKED STATE OF THE LPLL

Transfer Function of the Phase Detector

Input sinusoidal and VCO sinusoidal:

$$v_1(t) = V_{10}\, sin[\omega_1 t + \theta_1(t)] \qquad \text{and} \qquad v_2(t) = V_{20}\, cos[\omega_2 t + \theta_2(t)]$$

$$\therefore\ v_d(t) = v_1(t)\cdot v_2(t) = V_{10}V_{20}\, sin[\omega_1 t + \theta_1(t)]cos[\omega_2 t + \theta_2(t)]$$

$$= \frac{V_{10}V_{20}}{2} sin[\omega_1 t + \theta_1(t) - \omega_2 t - \theta_2(t)] - \frac{V_{10}V_{20}}{2} sin[\omega_1 t + \theta_1(t) + \omega_2 t + \theta_2(t)]$$

If the loop is locked, then $\omega_1 = \omega_2$ and

$$v_d(t) = \frac{V_{10}V_{20}}{2} sin[\theta_1(t) - \theta_2(t)] - \frac{V_{10}V_{20}}{2} sin[2\omega_1 t + \theta_1(t) + \theta_2(t)]$$

Ignoring the high-frequency terms gives,

$$v_d(t) \approx \frac{V_{10}V_{20}}{2} sin[\theta_1(t) - \theta_2(t)] = \frac{V_{10}V_{20}}{2} sin\theta_e(t) = K_d\, sin\theta_e(t) \approx K_d\, \theta_e(t)$$

if $\theta_e(t)$ is small.

$$K_d = \text{detector gain} = \frac{V_{10}V_{20}}{2}$$

$$\therefore\ v_d(t) \approx K_d\, \theta_e(t) \qquad \Rightarrow \qquad V_d(s) \approx K_d\, \Theta_e(s)$$

Transfer Function of the Phase Detector – Continued

Input signals when VCO output is a square wave:

$$v_1(t) = V_{10}\, sin[\omega_1 t + \theta_1(t)]$$

$$v_2(t) = V_{20}\, sgn[\omega_2 t + \theta_2(t)] = V_{20}\left[\frac{4}{\pi}cos[\omega_2 t + \theta_2(t)] + \frac{4}{3\pi}cos[3\omega_2 t + \theta_2(t)] + \cdots\right]$$

$$\therefore\ v_d(t) = v_1(t)\cdot v_2(t)$$

$$= V_{10}V_{20}\, sin[\omega_1 t + \theta_1(t)]\left[\frac{4}{\pi}cos[\omega_2 t + \theta_2(t)] + \frac{4}{3\pi}cos[3\omega_2 t + \theta_2(t)] + \cdots\right]$$

$$= \frac{4V_{10}V_{20}}{\pi}$$

$$\left[sin[\omega_1 t + \theta_1(t)]cos[\omega_2 t + \theta_2(t)] + \frac{1}{3}cos[\omega_2 t + \theta_2(t)]cos[3\omega_2 t + \theta_2(t)] + \cdots\right]$$

When the loop is locked,

$$v_d(t) = \frac{2V_{10}V_{20}}{\pi}\left[sin[\theta_1(t) - \theta_2(t)] + sin[2\omega_1 t + \theta_1(t) + \theta_2(t)] + \cdots\right]$$

$$\approx \frac{2V_{10}V_{20}}{\pi}\, sin\theta_e(t) = K_d\, sin\theta_e(t) \quad \rightarrow \quad v_d(t) \approx K_d\, \theta_e(t)$$

where the detector gain is $K_d = 2V_{10}V_{20}/\pi$ (a little better than sinusoidal inputs only)

The transfer function is $V_d(s) \approx K_d\, \Theta_e(s)$ or $\dfrac{V_d(s)}{\Theta_e(s)} = K_d$

VCO Transfer Function

The angular frequency of the VCO was expressed as,

$$\omega_2(t) = \omega_o + \Delta\omega_2(t) = \omega_o + K_o\, v_f(t)$$

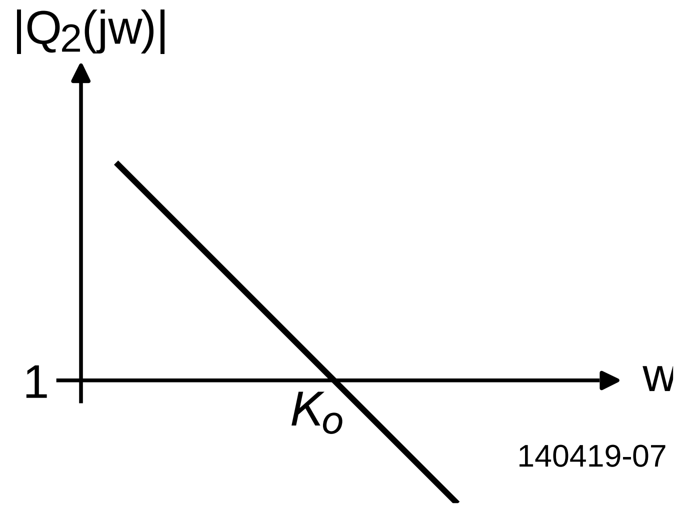
where K_o is the VCO gain in units of radians/sec or simply sec^{-1}.

However, what we want is the phase of the VCO output.

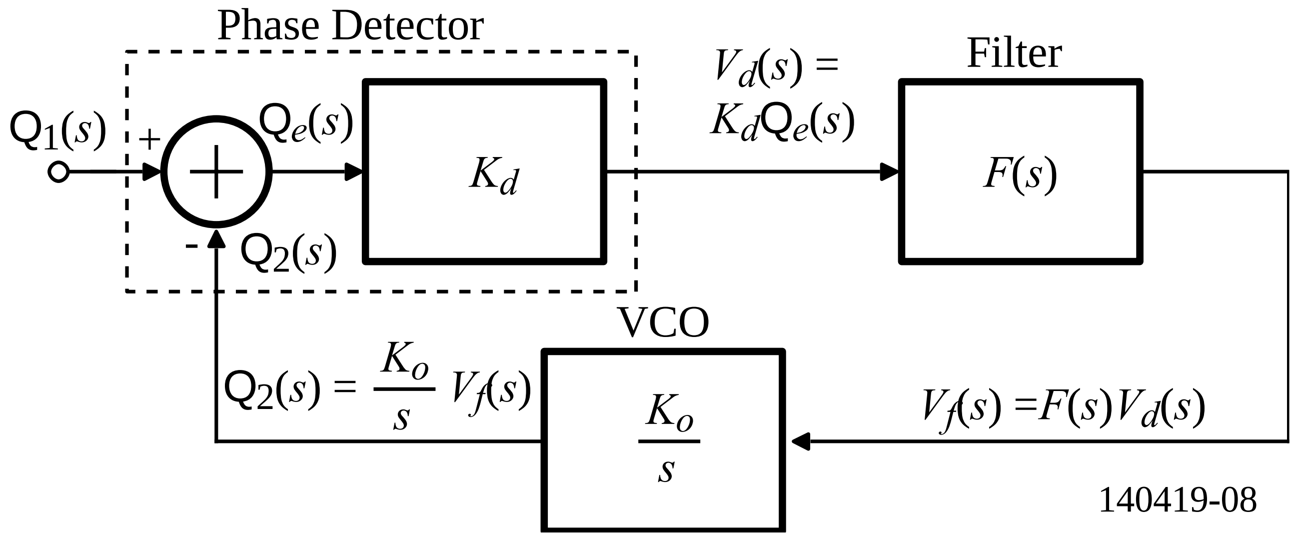
$$\therefore \quad \theta_2(t) = \int \Delta\omega_2\, dt = K_o \int v_f(t)dt$$

Taking the Laplace transform gives,

$$\Theta_2(s) = \mathrm{L}[\theta_2(t)] = \frac{K_o}{s} V_f(s) \quad \rightarrow \quad \frac{\Theta_2(s)}{V_f(s)} = \frac{K_o}{s}$$



Linear Model of the LPLL



Phase transfer function:

$$H(s) = \frac{\Theta_2(s)}{\Theta_1(s)} = ?$$

$$\Theta_2(s) = \frac{K_o}{s} V_f(s) = \frac{K_o}{s} F(s)V_d(s) = \frac{K_oK_d}{s} F(s)\Theta_e(s) = \frac{K_oK_d}{s} F(s)[\Theta_1(s) - \Theta_2(s)]$$

$$s\Theta_2(s) = K_oK_dF(s)[\Theta_1(s) - \Theta_2(s)] \quad \rightarrow \quad \Theta_2(s)[s + K_oK_dF(s)] = K_oK_dF(s)\Theta_1(s)$$

$$\therefore \quad H(s) = \frac{\Theta_2(s)}{\Theta_1(s)} = \frac{K_oK_dF(s)}{s + K_oK_dF(s)}$$

Also, $H_e(s) = \frac{\Theta_e(s)}{\Theta_1(s)} = 1 - H(s) = \frac{s}{s + K_oK_dF(s)}$

LPLL Transfer Function for Various Loop Filters

1.) Passive lag filter.

$$F(s) = \frac{1 + s\tau_2}{1 + s(\tau_1 + \tau_2)} \quad \rightarrow \quad H(s) = \frac{K_oK_d\left(\frac{1+s\tau_2}{\tau_1+\tau_2}\right)}{s^2 + s\left(\frac{1+K_oK_d\tau_2}{\tau_1+\tau_2}\right) + \frac{K_oK_d}{\tau_1+\tau_2}}$$

2.) The active lag filter.

$$F(s) = K_a\frac{1 + s\tau_2}{1 + s\tau_1} \quad \rightarrow \quad H(s) = \frac{K_oK_dK_a\left(\frac{1+s\tau_2}{\tau_1}\right)}{s^2 + s\left(\frac{1+K_oK_dK_a\tau_2}{\tau_1}\right) + \frac{K_oK_dK_a}{\tau_1}}$$

3.) The active PI filter.

$$F(s) = \frac{1 + s\tau_2}{s\tau_1} \quad \rightarrow \quad H(s) = \frac{K_oK_d\left(\frac{1+s\tau_2}{\tau_1}\right)}{s^2 + s\left(\frac{K_oK_d\tau_2}{\tau_1}\right) + \frac{K_oK_d}{\tau_1+\tau_2}}$$

Normalized Form of the Transfer Functions

The normalized form of the denominator of a second-order transfer function is

$$D(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

where ω_n is the natural frequency and ζ is the damping factor.

1.) Passive lag filter.

$$\omega_n = \sqrt{\frac{K_oK_d}{\tau_1+\tau_2}} \qquad \text{and} \qquad \zeta = \frac{\omega_n}{2}\left(\tau_2 + \frac{1}{K_oK_d}\right)$$

2.) Active lag filter.

$$\omega_n = \sqrt{\frac{K_oK_dK_a}{\tau_1}} \qquad \text{and} \qquad \zeta = \frac{\omega_n}{2}\left(\tau_2 + \frac{1}{K_oK_dK_a}\right)$$

3.) Active PI filter.

$$\omega_n = \sqrt{\frac{K_oK_d}{\tau_1}} \qquad \text{and} \qquad \zeta = \frac{\omega_n\tau_2}{2}$$

Normalized Phase Functions

1.) Passive lag filter.

$$H(s) = \frac{s\omega_n\left(2\zeta - \dfrac{\omega_n}{K_oK_d}\right) + \omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

2.) Active lag filter.

$$H(s) = \frac{s\omega_n\left(2\zeta - \dfrac{\omega_n}{K_oK_dK_a}\right) + \omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

3.) Active PI filter.

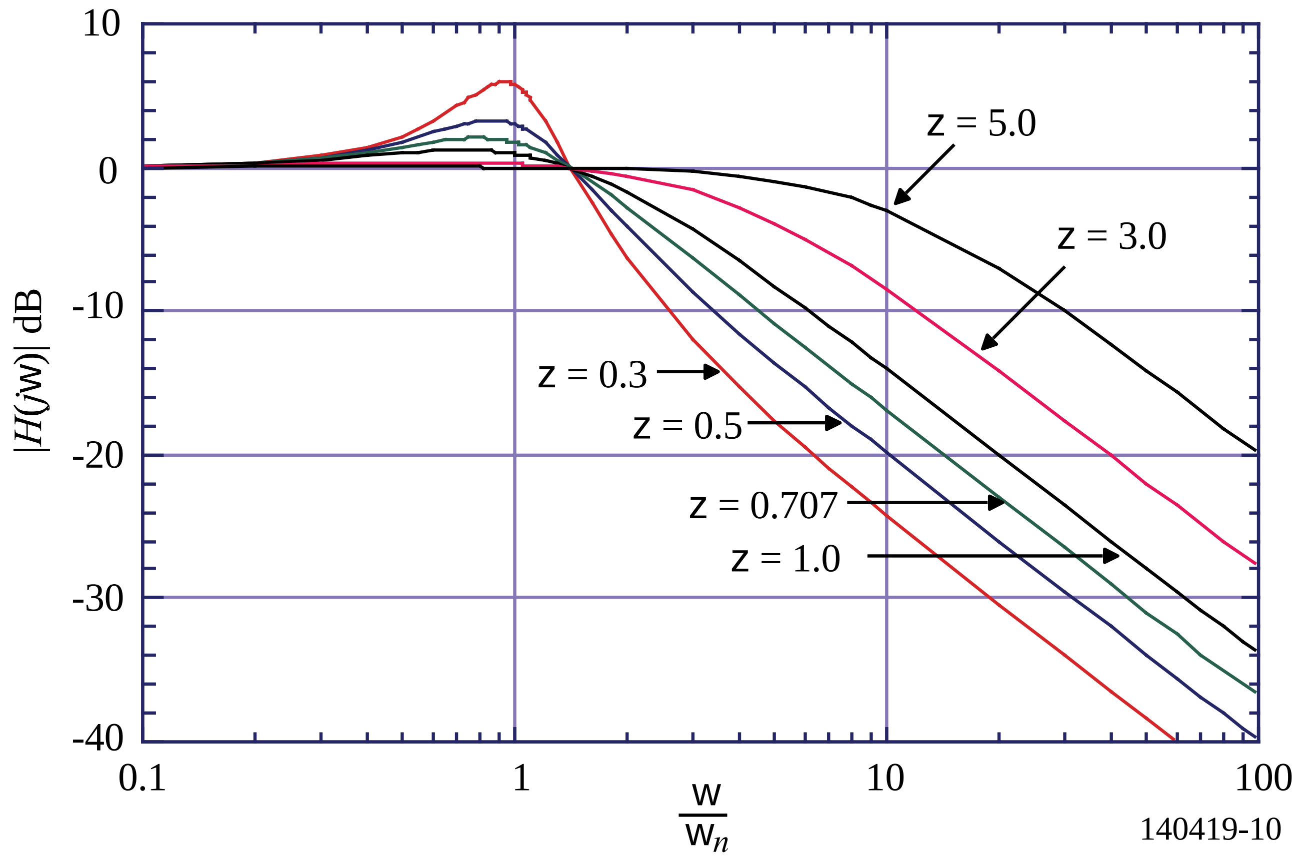
$$H(s) = \frac{2s\zeta\omega_n + \omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

If $K_dK_o >> \omega_n$ or $K_dK_oK_a >> \omega_n$ (high loop gain), then all of the above transfer functions simplify to,

$$H(s) \approx \frac{2s\zeta\omega_n + \omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2} \qquad \text{and} \qquad H_e(s) \approx \frac{s^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

Frequency Response of *H*(*s*)

Bode diagram:



Frequency response is normalized to ω/ω_n.

Phase Step Response

Assume that $\theta_1(t) = \Delta\Phi \cdot u(t)$ $\rightarrow$ $\Theta_1(s) = \dfrac{\Delta\Phi}{s}$

Phase error:

$$\Theta_e(s) = H_e(s)\,\frac{\Delta\Phi}{s} = \frac{\Delta\Phi s^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$\theta_e(t) = \mathrm{L}^{-1}[\Theta_e(s)] = \Delta\Phi\left(\cos\sqrt{1-\zeta^2}\,\omega_n t - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t\right)e^{-\zeta\omega_n t},\quad \zeta < 1$$

$$= \Delta\Phi(1 - \omega_n t)e^{-\zeta\omega_n t},\quad \zeta = 1$$

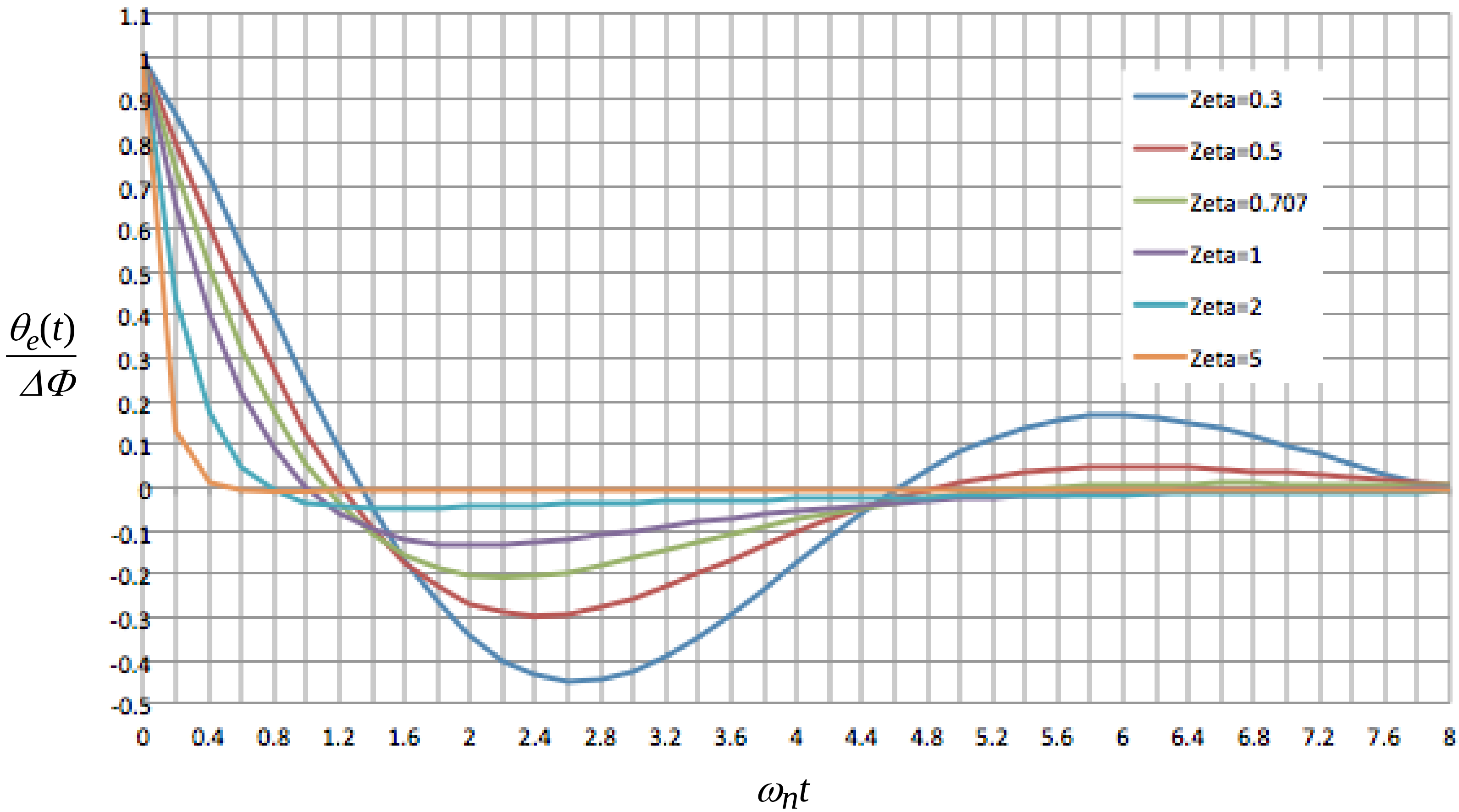
$$= \Delta\Phi\left(\cosh\sqrt{\zeta^2-1}\,\omega_n t - \frac{\zeta}{\sqrt{\zeta^2-1}}\sinh\sqrt{\zeta^2-1}\,\omega_n t\right)e^{-\zeta\omega_n t},\quad \zeta > 1$$

Steady state error:

$$\theta_e(\infty) = \lim_{s\to 0} s\Theta_e(s) = 0$$

Phase Step Response - Continued

Plot of the phase step response:



Frequency Step Response

Assume that $\omega_1(t) = \omega_o + \Delta\omega \cdot u(t)$

However, $\theta_1(t) = \Delta\omega \cdot t$ $\rightarrow$ $\Theta_1(s) = \dfrac{\Delta\omega}{s^2}$

Phase error:

$$\Theta_e(s) = H_e(s)\Theta_1(s) = H_e(s)\frac{\Delta\omega}{s^2} = \frac{\Delta\omega s^2}{s^2(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{\Delta\omega}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\theta_e(t) = \mathrm{L}^{-1}[\Theta_e(s)] = \frac{\Delta\omega}{\omega_n}\left(\frac{1}{\sqrt{1 - \zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t\right)e^{-\zeta\omega_n t}, \quad \zeta < 1$$

$$= \frac{\Delta\omega}{\omega_n}(\omega_n t)e^{-\zeta\omega_n t}, \quad \zeta = 1$$

$$= \frac{\Delta\omega}{\omega_n}\left(\frac{1}{\sqrt{\zeta^2-1}}\sinh\sqrt{\zeta^2-1}\,\omega_n t\right)e^{-\zeta\omega_n t}, \quad \zeta > 1$$

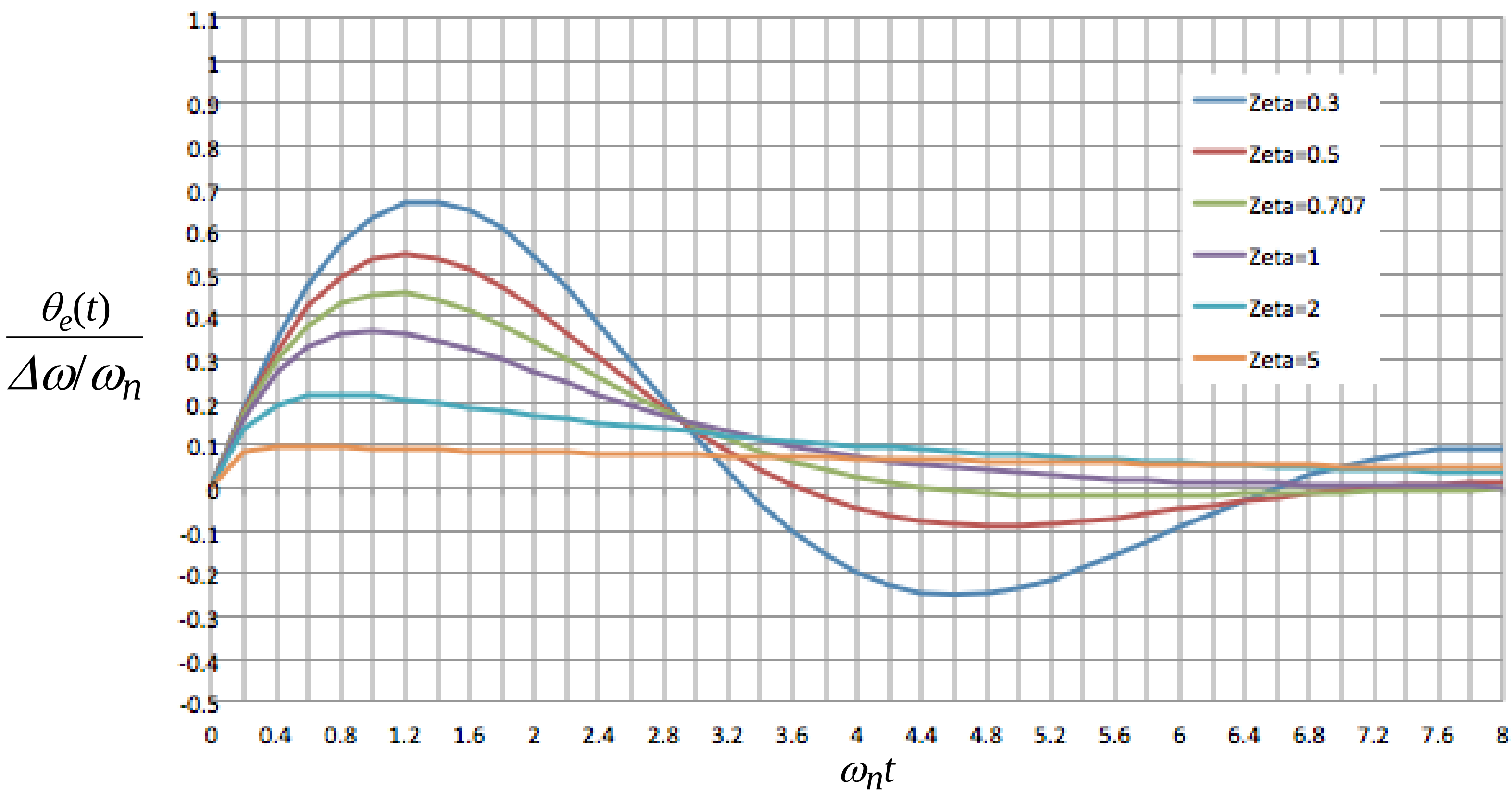
Steady state error:

$$\theta_e(\infty) = \lim_{s\to 0} s\Theta_e(s) = 0 \text{ (high gain loops, } K_dK_o \text{ or } K_dK_oK_a >> \omega_n)$$

$$\theta_e(\infty) = \frac{\Delta\omega}{K_dK_oF(0)} \text{ (low gain loops)}$$

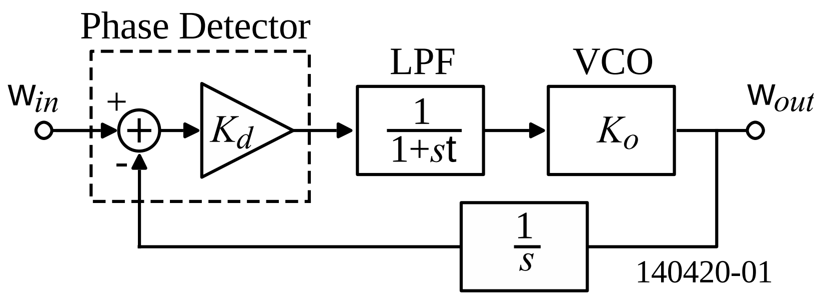
Frequency Step Response - Continued

Plot of the frequency step response:



Example 1 – Frequency Step Response

A linear model of a PLL is shown. (a.) Solve for the closed-loop transfer function of $\omega_{out}(s)/\omega_{in}(s)$. Compare this transfer function with the following transfer function and identify , H, ω_n, and ζ.



$$\frac{\omega_{out}(s)}{\omega_{in}(s)} = \frac{H\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

(b.) If $\zeta < 1$, the step response to $\omega_{in}(t) = \Delta\omega\cdot\mu(t)$ is given as

$$\omega_{out}(t) = H\Delta\omega\cdot\mu(t)\left[1 - \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\, sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \theta\right)\right] \text{where } \theta = sin^{-1}\sqrt{1-\zeta^2}$$

Assume that $K_v = K_oK_d = 63.58\text{x}10^3$ rads/sec. and $\tau = 8\mu$sec. If the output frequency is to be changed from 901 MHz to 901.2 MHz, how long does the PLL output frequency take to settle with 100 Hz of its final value? Simplify your analysis by assuming worst case conditions (i.e. Maximum value of $sin(x) = 1$).

Solution

Find the transfer function in terms of phase and then convert to frequency.

$$\theta_{out}(s) = \left(\frac{K_o}{s}\right)\left(\frac{1}{s\tau+1}\right) K_d[\theta_{in}(s)-\theta_{out}(s)] \;\rightarrow \theta_{out}(s)\left[1+\frac{K_v}{s(s\tau+1)}\right] = \frac{K_v}{s(s\tau+1)}\,\theta_{in}(s)$$

Example 1 - Continued

where $K_v = K_oK_d$. Solving for the phase transfer function gives,

$$\frac{\theta_{out}(s)}{\theta_{in}(s)} = \frac{\omega_{out}(s)}{\omega_{in}(s)} = \frac{K_v/\tau}{s^2 + (s/\tau) + (K_v/\tau)}$$

Therefore, $\boxed{\omega_n = \sqrt{\frac{K_v}{\tau}} = 89.148 \text{ krads/sec.}}$ $\boxed{\zeta = \frac{1}{2\sqrt{K_v\tau}} = 0.701}$ and $\boxed{H = 1}$

(b.) The frequency response can be written as

$$f_{out}(t) = 200\text{kHz}\left[1 - \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \theta\right)\right]\mu(t)$$

Setting $f_{out}(t_s) = 200\text{kHz} – 100\text{Hz}$, gives

$$200\text{x}10^3\text{-}100 = 200\text{x}10^3\left[1 - \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t_s} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t_s + \theta\right)\right]$$

This equation simplifies to the following assuming the value of the *sin* (x) is 1.

$$\frac{100\text{Hz}}{200\text{kHz}} = \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t_s} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t_s + \theta\right) \approx \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t_s}$$

$$e^{\zeta\omega_n t_s} = \frac{2000}{\sqrt{1-\zeta^2}} = 2800 \quad \rightarrow \quad t_s = \frac{1}{\omega_n\zeta} ln(2800) = 2\tau(\,7.9375) = \underline{\underline{127\mu\text{sec}}}.$$

Frequency Ramp Response

Assume that $\omega_1(t) = \omega_o + \Delta\dot{\omega} \cdot t$

However, $\theta_1(t) = \Delta\dot{\omega}\dfrac{t^2}{2}$ $\rightarrow$ $\Theta_1(s) = \dfrac{\Delta\dot{\omega}}{s^3}$

Phase error:

$$\Theta_e(s) = H_e(s)\frac{\Delta\dot{\omega}}{s^3} = \frac{\Delta\dot{\omega}\, s^2}{s^3(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{\Delta\dot{\omega}}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$\theta_e(t) = \mathrm{L}^{-1}[\Theta_e(s)] = \frac{\Delta\dot{\omega}}{\omega_n^2} - \frac{\Delta\dot{\omega}}{\omega_n^2}\left(\cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\zeta}{\sqrt{1+\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t\right)e^{-\zeta\omega_n t}, \quad \zeta < 1$$

$$= \frac{\Delta\dot{\omega}}{\omega_n^2} - \frac{\Delta\dot{\omega}}{\omega_n^2}(1 + \omega_n t)e^{-\zeta\omega_n t}, \quad \zeta = 1$$

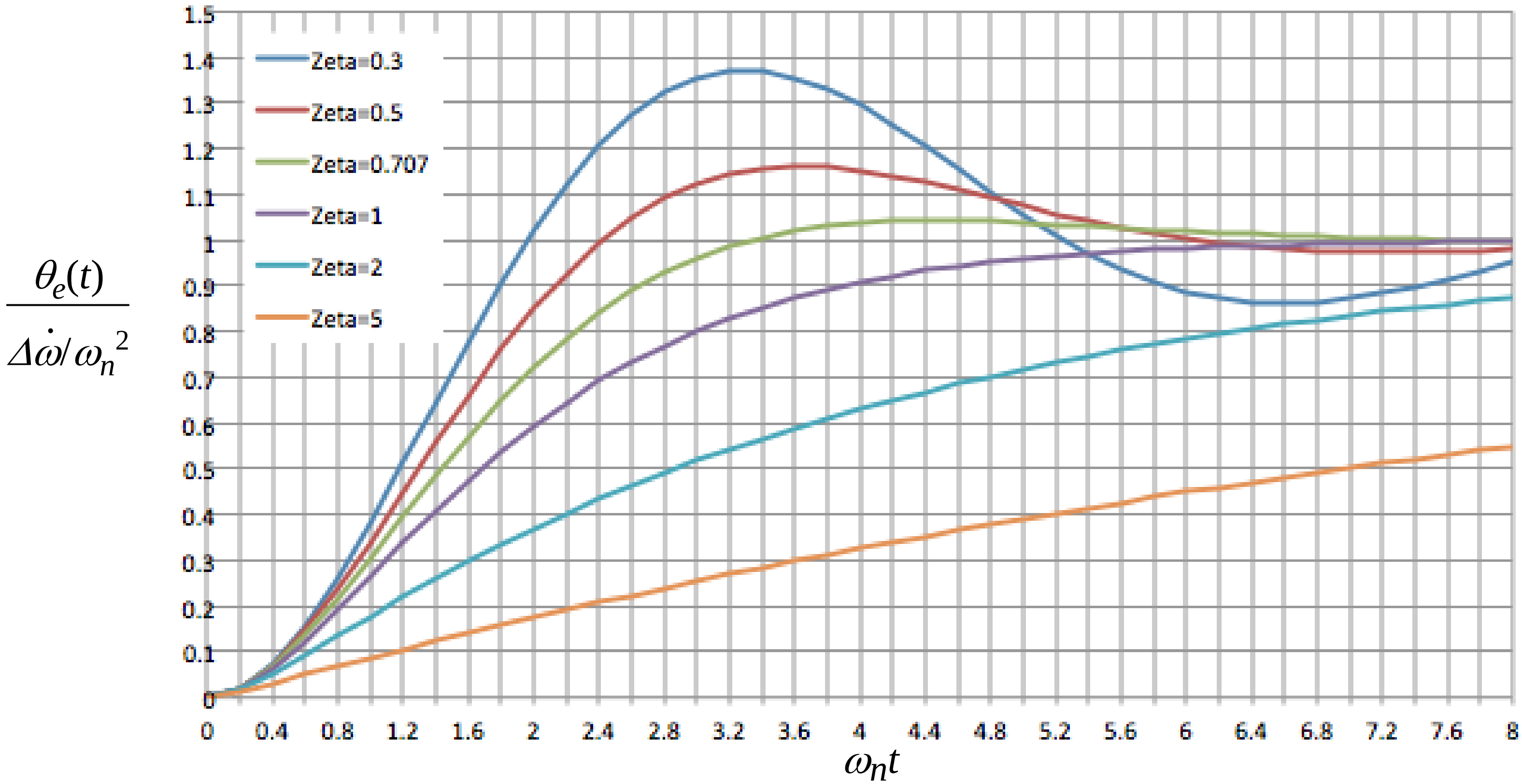
$$= \frac{\Delta\dot{\omega}}{\omega_n^2} - \frac{\Delta\dot{\omega}}{\omega_n^2}\left(\cosh\sqrt{\zeta^2-1}\,\omega_n t + \frac{\zeta}{\sqrt{\zeta^2-1}}\sinh\sqrt{\zeta^2-1}\,\omega_n t\right)e^{-\zeta\omega_n t}, \quad \zeta > 1$$

Steady state error:

$$\theta_e(\infty) = \lim_{s\to 0} s\Theta_e(s) = \frac{\Delta\dot{\omega}}{\omega_n^2} \text{ (High loop gain) } \quad \theta_e(\infty) = \frac{\Delta\dot{\omega}\, t}{K_d K_o F(0)^2} + \frac{\Delta\dot{\omega}}{\omega_n^2} \text{ (Low loop gain)}$$

Frequency Ramp Response

Plot of the frequency ramp step response:



THE ORDER OF A LPLL SYSTEM

Definition of Order

The number of roots in the denominator (poles) of the PLL transfer function determines the order.

Generally, the order of a PLL is one greater than the order of $F(s)$.

Implication of the order:

- Greater than 2 will be unstable unless corrected by zeros
- Less than 2 will have poor noise suppression.

First-Order PLL

A first–order PLL occurs when $F(s) = 1$. From previous results we have,

$$H(s) = \frac{\Theta_2(s)}{\Theta_1(s)} = \frac{K_oK_d}{s + K_oK_d} \qquad \text{Also, } H_e(s) = 1 - H(s) = \frac{s}{s + K_oK_d}$$

The –3dB bandwidth of $H(s)$ is K_oK_d.

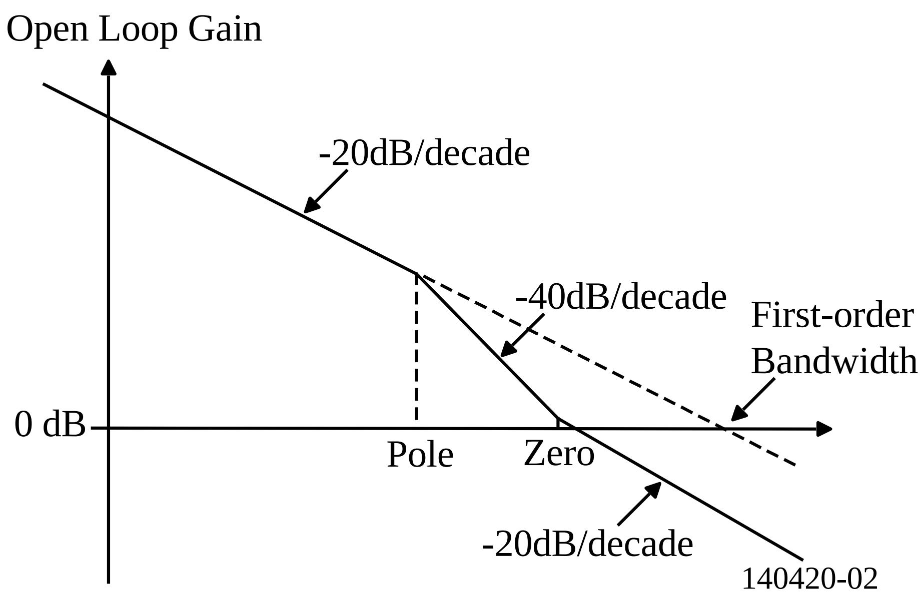
Comments:

- $F(s)$ causes the –3dB bandwidth to be reduced in higher-order systems which means that the first-order PLL has a wider bandwidth
- The hold range of the first-order PLL will be larger than for higher-order PLLs
- The first-order PLL will track the signal variations more quickly than higher-order PLLs
- The first-order PLL does not suppress noise superimposed on the input signal to the extent of higher-order PLLs.

Higher-Order PLLs

Comments:

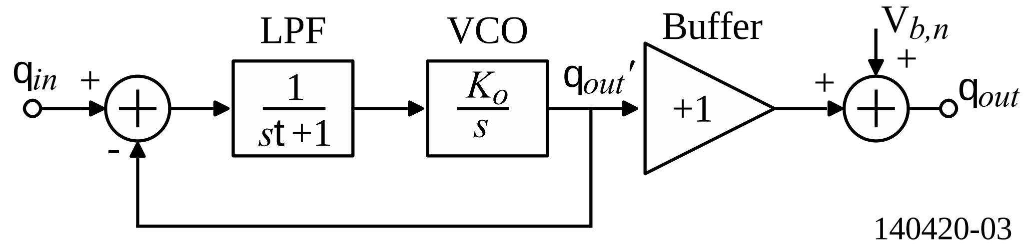
- Generally $F(s)$ has a pole and a zero in order to get better noise rejection without sacrificing speed.



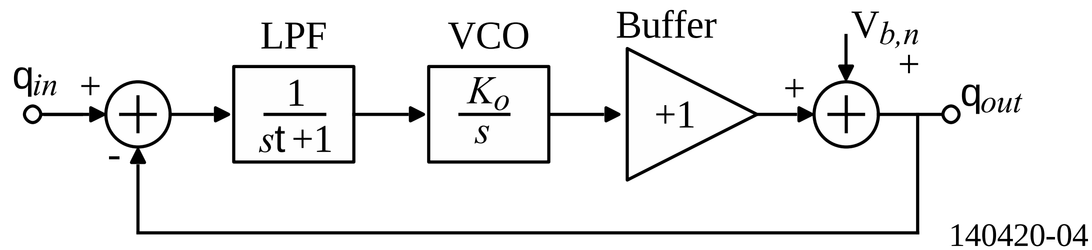
- If the phase shift of the open loop system is more than 90°, the stability of the loop may be poor (ζ is small).

Example 2- Buffering the PLL Output

In many practical applications, it is necessary to buffer the VCO output signal. Assume that the buffer has a voltage gain of 1 and its output is corrupted by an additive white noise voltage of $V_{b,n}$.



(a.) Find the output phase, $\theta_{out}(s)$, as a function of the input phase $\theta_{in}(s)$ and the output noise of the buffer, $V_{b,n}$, for the PLL shown with the buffer outside of the PLL loop. Give an approximate sketch for magnitude response of $\theta_{out}(j\omega)/V_{b,n}$ assuming $\zeta = 0.707$.



(b.) Find the output phase, $\theta_{out}(s)$, as a function of the input phase $\theta_{in}(s)$ and the output noise of the buffer, $V_{b,n}$, for the PLL shown with the buffer inside the PLL loop. Give an approximate sketch for magnitude response of $\theta_{out}(j\omega)/V_{b,n}$ assuming $\zeta = 0.707$.

(c.) Which of the two PLL architectures leads to an output spectrum with less noise assuming that the input and VCO are noise free? How would your answer change if the input signal to the PLL was noisy? Why?

Example 2 – Continued

Solution

(a.) $\theta_{out}'(s) = \dfrac{K_o}{s} F(s)K_d[\theta_{in}(s)-\theta_{out}'(s)] \rightarrow \quad \theta_{out}'(s) = \dfrac{K_vF(s)}{s+K_vF(s)}\,\theta_{in}(s)$

Substituting for $F(s)$ gives

$$\theta_{out}'(s) = \frac{K_v/\tau}{s^2+(s/\tau)+(K_v/\tau)}\,\theta_{in}(s) = \frac{\omega_n^2}{s^2+s2\zeta\omega_n+\omega_n^2}\,\theta_{in}(s)$$

$$\boxed{\theta_{out}(s) = \theta_{out}'(s) + V_{b,n} = \frac{\omega_n^2}{s^2+s2\zeta\omega_n+\omega_n^2}\,\theta_{in}(s) + V_{b,n}(s)}$$

(b.) $\theta_{out}(s) = \dfrac{K_o}{s} F(s)K_d[\theta_{in}(s)-\theta_{out}(s)] + V_{b,n}(s)$

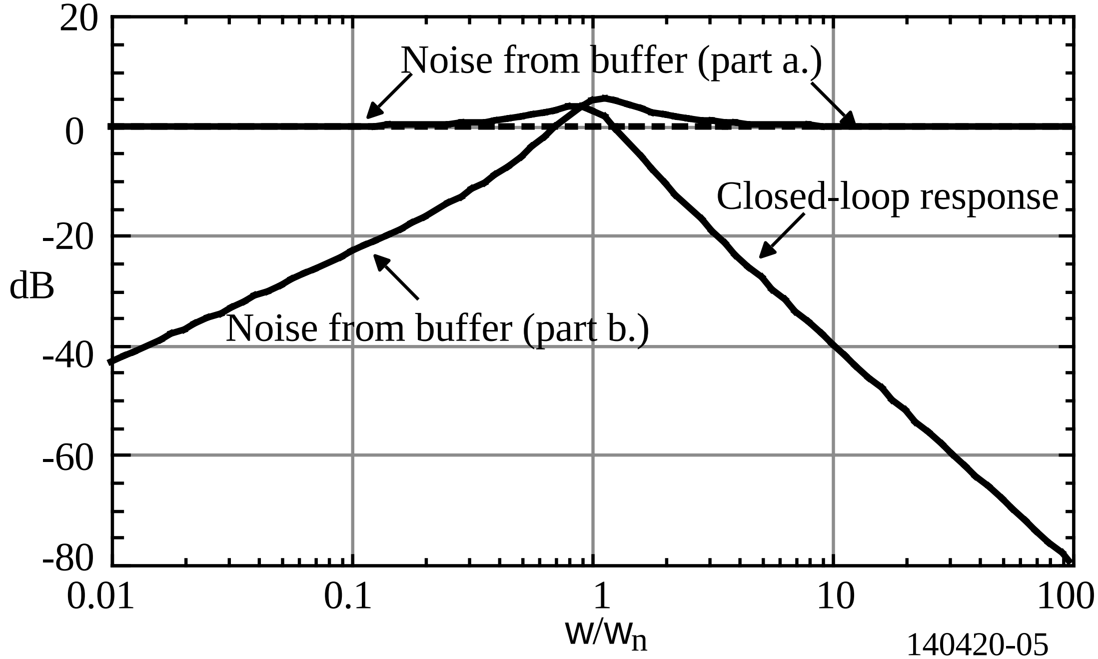
$$\theta_{out}(s)\left[1+\frac{K_vF(s)}{s}\right] = \frac{K_vF(s)}{s}\theta_{in}(s) + V_{b,n}(s)$$

$$\theta_{out}(s)=\frac{K_vF(s)}{s+K_vF(s)}\,\theta_{in}(s)+\frac{s}{s+K_vF(s)}\,V_{b,n}(s) = \frac{K_v/\tau}{s^2+(s/\tau)+(K_v/\tau)}\theta_{in}(s)+\frac{s^2+(s/\tau)}{s^2+(s/\tau)+(K_v/\tau)}\,V_{b,n}(s)$$

$$\boxed{\theta_{out}(s) = \frac{\omega_n^2}{s^2+s2\zeta\omega_n+\omega_n^2}\,\theta_{in}(s) + \frac{s^2+s2\zeta\omega_n}{s^2+s2\zeta\omega_n+\omega_n^2}V_{b,n}(s)}$$

Example 2 - Continued

The sketch for both parts (a.) and (b.) is shown below.



(c.) Obviously, part (b.) leads to an output spectrum with less noise. Part (a.) has the same noise contribution from the buffer regardless of the frequency. If the input is noisy then it will have a spectrum shown above similar to the closed-loop response. When the input noise is larger than the buffer noise, there is not much difference between the two architectures.

SUMMARY

- Unlocked state:
  - Hold range ($\Delta\omega_H$) – frequency range over which a PLL can statically maintain phase
  - Pull-in range ($\Delta\omega_P$) - frequency range within which a PLL will always lock
  - Pull-out range ($\Delta\omega_{PO}$) – dynamic limit for stable operation of a PLL
  - Lock range ($\Delta\omega_L$) - frequency range within which a PLL locks within one single-beat note between reference frequency and output frequency
- The order of a PLL is equal to the number of poles in the open-loop PLL transfer function